SECTION DE MATHÉMATIQUES EPFL

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TOPICS IN COMPLEX ANALYSIS
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# Solution for 'Topics in complex analysis'

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## H 11.1 (True or false?)

Prove the following statements or give a counterexample.

(i) The image of a simply connected domain under a non-constant holomorphic function is again a simply connected domain.

(ii) The image of a simply connected domain under an injective holomorphic function is again a simply connected domain.

(iii) The complex plane is not biholomorphically equivalent to any simply connected domain $G \subsetneq \mathbb{C}$.

(iv) The set $\mathbb{C} \setminus B_r(z_0)$ with $z_0 \in \mathbb{C}$ and $r > 0$ is simply connected.

## Solution H 11.1:

a) False. The complex exponential maps the simply connected domain $\mathbb{C}$ to the non-simply connected domain $\mathbb{C} \setminus \{0\}$.

b) True. The open mapping theorem implies that $f(G)$ is a domain. By Lemma 7.1 the inverse $f^{-1} : f(G) \to G$ is holomorphic, so in particular continuous. Let $\gamma \subset f(G)$ be a continuous closed curve. Then $t \mapsto f^{-1}(\gamma(t))$ is a continuous closed curve in $G$. Hence there exists a continuous map $H : [0,1]^2 \to G$ and $z_0 \in G$ such that $H(0,t) = f^{-1}(\gamma(t))$, $H(1,t) = z_0$ and $H(s,0) = H(s,1)$ for all $s,t \in [0,1]$. The continuous map $S = f \circ H : [0,1]^2 \to f(G)$ then satisfies $S(0,t) = \gamma(t)$, $S(1,t) = f(z_0)$ and $S(s,0) = S(s,1)$ for all $s,t \in [0,1]$. Hence $\gamma$ can be contracted to a point using $S$.

c) True. If it were biholomorphically equivalent to a simply connected domain $G \subsetneq \mathbb{C}$, then by the Riemann mapping theorem it would also be biholomorphically equivalent to the unit disc, which contradicts Liouville's theorem.

d) False. Otherwise the function $z \mapsto \frac{1}{z-z_0}$ would have a primitive on $\mathbb{C} \setminus B_r(z_0)$. But this contradicts the fact that $\int_{\partial B_{2r}(z_0)} \frac{1}{z-z_0}\, dz \neq 0$.

$\square$

## H 11.2 (Schwarz lemma on simply connected domains)

Let $G \subsetneq \mathbb{C}$ be a simply connected domain. Given $a \in G$ we denote by $\mathrm{Hol}_a(G)$ the set of holomorphic functions $f : G \to G$ such that $f(a) = a$. Prove that $|f'(a)| \leq 1$ for all $f \in \mathrm{Hol}_a(G)$. Moreover, show that $f \in \mathrm{Hol}_a(G)$ is bijective if and only if $|f'(a)| = 1$.
**Hint**: Use the function given by the Riemann mapping theorem.

## Solution H 11.2:

Let $g : G \to B_1(0)$ be the biholomorphic map given by the Riemann mapping theorem. Let $z_0 = g(a) \in B_1(0)$ and denote by $\varphi_{z_0} : B_1(0) \to B_1(0)$ the biholomorphic map

$$\varphi_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}$$

from Exercise H 9.2. Upon replacing $g$ with $\varphi_{z_0} \circ g$ we may assume that $g(a) = 0$. Then $h := g \circ f \circ g^{-1} : B_1(0) \to B_1(0)$ is holomorphic and satisfies $h(0) = 0$. Hence by the standard Schwarz lemma we deduce that $|h'(0)| \leq 1$. The chain rule gives

$$|g'(a)| \cdot |f'(a)| \cdot |(g^{-1})'(0)| = |g'(f(g^{-1}(0))) \cdot f'(g^{-1}(0)) \cdot (g^{-1})'(0)| = |h'(0)| \leq 1.$$

Since $(g^{-1})'(0) = (g^{-1})'(g(a)) = \frac{1}{g'(a)} \neq 0$, we infer that $|f'(a)| \leq 1$ as claimed.
If $f$ is bijective, then the Schwarz lemma (applied to $h$ and its inverse) implies that $h$ is a rotation, so that in the above estimate we have equality. Thus as before we conclude that $|f'(a)| = 1$. On the other hand, if $|f'(a)| = 1$, then the chain rule implies again that $|h'(0)| = 1$, so that $h$ is a rotation. Hence $f$ has to be bijective.

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**H 11.3 (Singularities of injective holomorphic maps)**
Let $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic and injective. Prove that either $z_0$ is a removable singularity and the continuous extension to $z_0$ is still injective or $z_0$ is a pole of first order.
**Hint**: Rule out an essential singularity using Picard's great theorem.

**Solution H 11.3:**
Assume first that $z_0$ is a removable singularity and assume by contradiction that the extension is not injective. This means there exists $z_1 \in U \setminus \{z_0\}$ such that $f(z_0) = f(z_1)$. Let $r > 0$ be such that $B_r(z_0) \cap B_r(z_1) = \emptyset$ and $B_r(z_0), B_r(z_1) \subset U$. Since the extension is still holomorphic on $U$, we know that $f(B_r(z_0))$ and $f(B_r(z_1))$ are open sets since the function $f$ is not constant in either ball. Thus their intersection is also open and non-empty since $f(z_0) \in f(B_r(z_0)) \cap f(B_r(z_1))$. Hence the intersection contains more than one element. This contradicts the injectivity of $f$ on $U \setminus \{z_0\}$.
Next we assume that $z_0$ is not a removable singularity. Assume by contradiction that it is an essential singularity. Then for every $r > 0$ small enough the set $f(B_r(z_0) \setminus \{z_0\})$ is open by the open mapping theorem. Moreover, also the set $f(U \setminus \overline{B_r(z_0)})$ is open and nonempty. Due to injectivity they are disjoint. Hence $f(B_r(z_0) \setminus \{z_0\})$ cannot even be dense in $\mathbb{C}$, contradicting Picard's great theorem (or the much simpler Casorati-Weierstrass theorem).
We conclude that $f$ has a pole at $z_0$. This implies that there exists a ball $B_s(z_0)$ such that $f(z) \neq 0$ for all $z \in B_s(z_0) \setminus \{z_0\}$. Hence the function $g(z) = 1/f(z)$ on $B_s(z_0) \setminus \{z_0\}$ is holomorphic, injective, and has a removable singularity at $z_0$. Thus we can extend it to a holomorphic and injective (cf. the first part of the proof) function $g : B_s(0) \to \mathbb{C}$ such that $g(z_0) = 0$. Then Lemma 7.1 yields that $g'(z_0) \neq 0$, so that $g$ has a simple zero at $z_0$. Consequently $f$ has a simple pole at $z_0$.

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**H 11.4 (Rigidity of biholomorphic maps)**
a) Let $f : B_1(0) \to B_1(0)$ be a biholomorphic map such that $f(a) = a$ and $f(b) = b$ for two distinct $a, b \in B_1(0)$. Prove that $f(z) = z$ for all $z \in B_1(0)$.

b) Show that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and injective if and only if $f(z) = az + b$ for some $a, b \in \mathbb{C}$ and $a \neq 0$.
**Hint**: Try to apply Exercise H 11.3.

c) Show that $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is holomorphic and injective if and only if $f(z) = az$ or $f(z) = az^{-1}$ for some $a \in \mathbb{C} \setminus \{0\}$.

d) Let $G \subsetneq \mathbb{C}$ be a simply connected domain and let $f \in \mathrm{Hol}_a(G)$ (cf. Exercise H 11.2) be biholomorphic. Show that $f'(a) \in (0, +\infty)$ implies that $f(z) = z$ for all $z \in G$.

**Solution H 11.4:**

a) Consider the biholomorphic function $g = \varphi_a \circ f \circ \varphi_a^{-1} : B_1(0) \to B_1(0)$. We already know from Exercise H 9.2 that $g$ has the representation

$$g(z) = c\frac{z - z_0}{1 - \overline{z_0}z}$$

for some $z_0 \in B_1(0)$ and $c \in \partial B_1(0)$. Since $g(0) = 0$ it follows that $z_0 = 0$. Moreover, note that $g(\varphi_a(b)) = \varphi_a(b)$. Since $\varphi_a(b) \neq 0$, this implies $c = 1$. Consequently $g(z) = z$ for all $z \in B_1(0)$, and therefore also $f(z) = z$ for all $z \in B_1(0)$.

b) Clearly non-constant affine functions are injective and holomorphic. To prove the converse, consider the injective, holomorphic function $z \mapsto f(\frac{1}{z}) : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, which has a global Laurent-series expansion

$$f(\tfrac{1}{z}) = \sum_{n=0}^{\infty} c_n z^{-n}.$$

By Exercise H 11.3 we know that $c_n = 0$ for all $n \geq 2$. Hence $f(\frac{1}{z}) = c_0 + c_1 z^{-1}$ for all $z \in \mathbb{C} \setminus \{0\}$. This implies that $f(z) = c_1 z + c_0$ is affine, and due to injectivity $c_1 \neq 0$.

c) All functions of the form $f(z) = az$ or $f(z) = az^{-1}$ with $a \neq 0$ are injective, holomorphic on $\mathbb{C} \setminus \{0\}$, and map to $\mathbb{C} \setminus \{0\}$, so it remains to prove the converse statement.
According to Exercise H 11.3, $f$ has either a removable singularity or a pole of first order at 0. If the singularity is removable then according to Exercise H 11.3 the continuous extension would be a holomorphic and injective function $f : \mathbb{C} \to \mathbb{C}$, so that b) implies that $f(z) = az + b$. If $b \neq 0$ then $f(z_0) = 0$ for some $z_0 \neq 0$, which contradicts the assumption that $f(\mathbb{C} \setminus \{0\}) \subset \mathbb{C} \setminus \{0\}$. Hence $f(z) = az$ for some $a \neq 0$. If the singularity is a pole of first order, then the holomorphic and injective function $1/f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ has a removable singularity at 0, so that by the previous argument $1/f(z) = az$ for some $a \neq 0$. Hence $f(z) = a^{-1}z^{-1}$ for all $z \in \mathbb{C} \setminus \{0\}$, as claimed.

d) Since $f$ is assumed to be biholomorphic and $f'(a) \in (0, +\infty)$, we deduce from Exercise H 11.2 that $f'(a) = 1$. Repeating the chain rule argument from the solution of Exercise H 11.2 (without the modulus), we conclude that $h : B_1(0) \to B_1(0)$ is a rotation with $h'(0) = f'(a) = 1$, so that $h(z) = z$ for all $z \in B_1(0)$. Hence also $f(z) = z$ for all $z \in G$. $\square$

**H 11.5 (Examples of the Riemann mapping theorem)**
In this exercise we provide biholomorphic maps $f : G \to B_1(0)$ for some special sets $G$.

a) Show that the map $z \mapsto \frac{z-i}{z+i}$ is biholomorphic from $\mathbb{H}_+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ to $B_1(0)$.

b) Find a biholomorphic map $f : \mathbb{C} \setminus (-\infty, 0] \to B_1(0)$.

**Solution H 11.5:**
a) Clearly the map is holomorphic and well-defined, as $-i \notin \mathbb{H}_+$ and

$$|z - i| < |z + i| \iff zi - \overline{z}i < -zi + \overline{z}i \iff \operatorname{Im}(z) > 0.$$

Hence it only remains to show that the map is bijective. To this end, note that the inverse is given by $z \mapsto i\,\frac{z+1}{1-z}$.

b) Since $G := \mathbb{C}\setminus(-\infty, 0]$ is star-shaped, it is in particular simply connected. Hence there exists a holomorphic square-root $\sqrt{\cdot} : G \to \mathbb{C}$. It is injective and we claim that $\operatorname{Re}(\sqrt{z}) > 0$ for all $z \in G$. Indeed, taking the principal branch of the logarithm such that $\arg\log(z) \in (-\pi, \pi)$ for all $z \in G$, the square-root can be written as $\sqrt{z} = \exp(\frac{1}{2}\log(z))$, so that $\arg(\sqrt{z}) \in (-\pi/2, \pi/2)$. Moreover, for any $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$ we know that $\arg(z) \in (-\pi/2, \pi/2)$, so that $z^2 \in G$. Consequently the map $\sqrt{\cdot} : G \to \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ is biholomorphic. As a next step we note that by the rotation $z \mapsto iz$ we can map the set $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ biholomorphically to $\mathbb{H}_+$ (recall that multiplication by $i$ represents a counterclockwise rotation by 90°). Then we can use a) to map $\mathbb{H}_+$ biholomorphically to $B_1(0)$. The composition reads

$$f(z) = \frac{i\sqrt{z} - i}{i\sqrt{z} + i} = \frac{\sqrt{z} - 1}{\sqrt{z} + 1}.$$

$\square$